LexiFi

Stochastic Local Volatility models for Inflation

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Abstract

We present an algorithm based on the particle method, a recent work on stochastic local volatility models calibration, that allows to jointly calibrate a stochastic local volatility model on both the smile and its dynamics. We apply this methodology to inflation indices, as for inflation, both the smile and its dynamics are quoted on the main markets through index option and year over year option quotes. The algorithm can also be used on FX markets when one touch option prices are available. We provide numerical results and compare the execution time overcost to match the smile dynamics of this novel approach to the usual particle method.

1 Introduction

Inflation indices are often modeled as an exchange rate between "real" and "nominal" currency. Classical index models are therefore of the form:

$$\frac{dI(t)}{I(t)} = [n(t) - r(t)]dt + \sigma(t)dW(t)$$

where $n(t)$ (resp. $r(t)$) is the nominal (resp. real) short rate at t, and σ is a $(\mathcal{F}_t)$-measurable process.

Contrary to the usual FX market, there exist quoted option prices both on the index (Zero-Coupon options) and on the year over year (YoY). In a modeling perspective, we therefore have access to information about the smile and its dynamics, and a simple smile model (e.g. Local Volatility or Heston model) cannot match the call prices for both option kinds. However, Local Volatility (LV) models and Stochastic Volatility (SV) models have different smile dynamics. Therefore, mixing both models provides one more parameter (the importance of the Stochastic or Local Volatility) that allows to fit 1) perfectly the smile thanks to the Local Volatility part and 2) fit also the smile dynamics using this parameter. We present in this paper a method that allows calibration of any Stochastic Local Volatility (SLV) model (given that we are able to calibrate the pure stochastic part) to both the Zero-Coupon and YoY option prices.

2 The general SLV model

We write our generic SLV model:

$$\begin{aligned}\frac{dI(t)}{I(t)} &= [n(t) - r(t)]dt + \sigma(t)L(t, I(t))dW(t) \\ d\sigma(t) &= a(t, \sigma(t))dt + b(t, \sigma(t))dB(t) \\ dW(t)dB(t) &= \rho dt\end{aligned}$$

with $L \in \mathbb{R}^{\mathbb{R}^+ \times \mathbb{R}}$ a deterministic function, called the leverage surface.

We assume that interest rates are deterministic.

For the sake of simplicity, we rewrite the model:

$$\begin{aligned}\frac{d\tilde{I}(t)}{\tilde{I}(t)} &= \sigma(t)L(t, I(t))dW(t) \\ I(t) &= F(0, t)\tilde{I}(t)\end{aligned}$$

with

$$F(0, t) = I(0)\frac{P^r(0, t)}{P^n(0, t)}$$

where P^n (resp. P^r) is the nominal (resp. real) Zero-Coupon price.

3 Leverage surface Calibration

In this section, we show how to find the leverage surface that fit the Zero-Coupon option prices for given stochastic volatility parameters a and b. Later, we will show how to adapt these SV parameters a and b to handle the smile dynamics.

3.1 Main idea

We use the so called "markovian projection" technique (see for instance [Pit07] for more details) to express the leverage surface as a function of the local volatility.

Let X_1 and X_2 defined by

$$\begin{aligned}dX_i(t) &= b_i(t)dW_i(t) \\ X_i(t) &= X_0\end{aligned}$$

Then,

$$\forall (t, x) \in (\mathbb{R}^+ \times \mathcal{X}), \mathbb{E}[b_1(t)^2 | X_1(t) = x] = \mathbb{E}[b_2(t)^2 | X_2(t) = x] \Rightarrow \forall t \in \mathbb{R}^+, X_1(t) \stackrel{\mathcal{L}}{=} X_2(t)$$

In particular, the option prices are equal in both models.

Back to the SLV case: we set $b_1(t) = \sigma(t)L(t, X_1(t))X_1(t)$ and $b_2(t) = \Sigma(t, X_2(t))X_2(t)$.

and

$$L(t,x)^2 = \frac{\Sigma(t,x)^2}{\mathbb{E}[\sigma(t)^2|X_1(t) = x]} \tag{3.1}$$

making the call prices on the Local Volatility and on the SLV models identical.

We recall that we can match the call prices in a Local Volatility model thanks to the Dupire equation [Dup94] (without rates in our case):

$$\Sigma(t,K)^2 = \frac{2\partial_t C_X(t,K)}{K^2\partial_K^2 C_X(t,K)} \tag{3.2}$$

where $C_X(t,K) = \mathbb{E}[(X-K)^+]$.

These remarks lead to the classical SLV calibration method:

1. Calibrate the parameters of σ.
2. Calibrate the local volatility Σ on the market call prices using (3.2).
3. Set L according to (3.1).

Computing $\mathbb{E}[\sigma(t)^2|X(t) = x]$ remains however a difficult task that we investigate in the next section.

3.2 Computation of the conditional variance expectation

Let us consider a generic method to compute $\mathbb{E}[\sigma(t)^2|X(t) = x]$ based on the particle method developed in [JG11].

First, we show that:

$$\mathbb{E}[f(X)|Y = y] = \frac{\mathbb{E}[f(X)\delta(Y-y)]}{\mathbb{E}[\delta(Y-y)]} \tag{3.3}$$

<u>Proof</u>:

$$\begin{aligned}\mathbb{E}[f(X)\delta(Y-y)] &= \int_{\mathcal{X}}\int_{\mathcal{Y}} f(u)\delta(v-y)f_{X,Y}(u,v)dvdu \\ &= \int_{\mathcal{X}} f(u)f_{X,Y}(u,y)du\end{aligned}$$

In particular, with $f = 1$, we have

$$\begin{aligned}\mathbb{E}[\delta(Y-y)] &= \int_{\mathcal{X}} f_{X,Y}(u,y)du \\ \Rightarrow \frac{\mathbb{E}[f(X)\delta(Y-y)]}{\mathbb{E}[\delta(Y-y)]} &= \frac{\int_{\mathcal{X}} f(u)f_{X,Y}(u,y)du}{\int_{\mathcal{X}} f_{X,Y}(u,y)du} = \mathbb{E}[f(X)|Y=y]\end{aligned}$$

The idea of the particle method is to compute the conditional expectation using formulation (3.3) with a Monte Carlo method and an approximation function for δ.

We introduce the following variables:

- The simulation dates $t_0 = 0 < t_1 < ... < t_n = T$.
- The matrices $(\bar{X}, \bar{\sigma}) \in \mathcal{M}_{n+1,m}(\mathbb{R})^2$ so that $\bar{X}_i^j$ (resp. $\bar{\sigma}_i^j$) is the j-th realization of $X(t_i)$ (resp. $\sigma(t_i)$) ($\bar{X}^j$ is also called the j-th particle).
- $(x_l)_{l\in\mathcal{L}}$ a discrete set of strikes on which we will compute the leverage surface.
- $(\delta_{i,m})_{i\in[0,n]}$ a family of regularizing kernels, so that $\forall i \in [0, n], \delta_{i,m} \underset{m\to+\infty}{\longrightarrow} \delta$.

The algorithm for the particle method is then:

1. Set $L(0, x) := \frac{\Sigma(0,x)}{\sigma(0)}$.
2. Set $\bar{X}_0^j := X(0), \bar{\sigma}_0^j = \sigma(0)$.
3. $\forall i \in [1; n]$
   (a) simulate $\bar{X}_i^j, \bar{\sigma}_i^j$ using $\bar{X}_{i-1}^j$, $L(t_{i-1}, \bar{X}_{i-1}^j)$ and $\bar{\sigma}_{i-1}^j$.
   (b) $\forall l \in \mathcal{L}$, set
   $$L(t_i, x_l)^2 := \Sigma(t_i, x_l)^2 \frac{\sum_{j=0}^{m-1} \delta_{i,m}(\bar{X}_i^j - x_l)}{\sum_{j=0}^{m-1} (\bar{\sigma}_i^j)^2 \delta_{i,m}(\bar{X}_i^j - x_l)}$$
   (c) interpolate $L(t_i, \cdot)$ with $(x_l, L(t_i, x_l))_{l\in\mathcal{L}}$.

For $\delta_{i,m}$, we choose

$$\begin{aligned}
\delta_{i,m}(x) &= \frac{1}{h_{i,m}} K\left(\frac{x}{h_{i,m}}\right) \\
K(x) &= \frac{15}{16}(1 - x^2)^2 \mathbf{1}_{\{|x|\leq 1\}} \\
h_{i,m} &= \frac{3}{2} m^{-1/5} \Sigma(t_i, X(0)) \sqrt{max\left(t_i, \frac{1}{4}\right)}
\end{aligned}$$

An advantage of using such a method is that during the computation of L, we can compute simultaneously any expectations of $f((X(t_i))_{i\in[1,n]})$, and in particular prices of YoY options:

$$\mathbb{E}[f((X(t_i))_{i\in[1,n]})] \approx \frac{1}{m} \sum_{j=0}^{m-1} f(\bar{X}^j)$$

3.3 Back to inflation

In this section, we specify briefly the calibration in the case of our SLV inflation model. The first step is to set $X := \tilde{I}$. We must then compute the function C_X, thanks to the Zero-Coupon option prices.

The payoff of the ZC call with strike K maturing at t is $\left(\frac{I(t)}{I(0)} - (1+K)^t\right)^+$. Let $C(t,K)$ be its price.

Defining $F_0 = \frac{F}{I(0)} = \frac{P^r}{P^n}$, we have

$$\begin{aligned} C(t,K) &= P^n(0,t)\mathbb{E}\left[\left(\frac{I(t)}{I(0)} - (1+K)^t\right)^+\right] \\ &= P^n(0,t)\frac{F(0,t)}{I(0)}\mathbb{E}\left[\left(\tilde{I}(t) - (1+K)^t\frac{I(0)}{F(0,t)}\right)^+\right] \\ &= P^n(0,t)F_0(0,t)\mathbb{E}\left[\left(\tilde{I}(t) - \frac{(1+K)^t}{F_0(0,t)}\right)^+\right] \\ &= P^n(0,t)F_0(0,t)C_X\left(t, \frac{(1+K)^t}{F_0(0,t)}\right) \end{aligned}$$

Similarly, we express the prices of the YoY calls, and show how to compute them during the particle method algorithm. We consider the YoY caplet with strike K, maturing at $t = t_{i_1} > 1$. Let i_0 such as $t_{i_0} = t-1$. We define $F_{YoY}(0,t) = \frac{F(0,t)}{F(0,t-1)}$

The payoff of this option is $\left(\frac{I(t)}{I(t-1)} - 1 - K\right)^+$, and its price C_{YoY} verifies:

$$\begin{aligned} C_{YoY}(t,K) &= P^n(0,t)\mathbb{E}\left[\left(\frac{I(t)}{I(t-1)} - 1 - K\right)^+\right] \\ &= P^n(0,t)F_{YoY}(0,t)\mathbb{E}\left[\left(\frac{\tilde{I}(t)}{\tilde{I}(t-1)} - \frac{1+K}{F_{YoY}(0,t)}\right)^+\right] \end{aligned}$$

$$\Rightarrow C_{YoY}(t,K) \approx P^n(0,t)F_{YoY}(0,t)\frac{1}{m}\sum_{j=0}^{m-1}\left(\frac{X^j_{i_1}}{X^j_{i_0}} - \frac{1+K}{F_{YoY}(0,t)}\right)^+ \tag{3.4}$$

4 Calibration of the stochastic volatility

The next step consists in adapting the stochastic volatility parameters to match the YoY caplet prices.

First, we calibrate the pure SV model on Zero-Coupon call prices:

$$\begin{aligned} d\tilde{I}(t) &= \sigma(t)\tilde{I}(t)dW(t) \\ d\sigma(t) &= a(t,\sigma(t))dt + b(t,\sigma(t))dB(t) \end{aligned}$$

We then specify two functions $\bar{a}$, $\bar{b}$ that verify $\forall (t,x) \in (\mathbb{R}^+ \times \mathcal{X})$:

- $\bar{b}(t,x,0) = 0$.
- $\bar{a}(t,x,1) = a(t,x)$.
- $\bar{b}(t,x,1) = b(t,x)$.
- $\bar{a}(t,x,\cdot)$ is monotonous.
- $\bar{b}(t,x,\cdot)$ is strictly increasing.

We introduce a new parameter $\lambda > 0$, usually called the mixing weight or the mixing fraction (see [Cla11] or [TF10]), that allows to define a new range of models:

$$\frac{d\tilde{I}_\lambda(t)}{\tilde{I}_\lambda(t)} = \sigma_\lambda(t) L_\lambda(t, I_\lambda(t)) dW(t)$$
$$d\sigma_\lambda(t) = a(t, \sigma(t), \lambda)dt + b(t, \sigma(t), \lambda) dB(t)$$

First condition means that the model is the pure LV for $\lambda = 0$. Second and third conditions mean that the model is SV dominated for $\lambda = 1$, where the leverage surface only removes the pure SV calibration error. Last two conditions imply that there is a "monotonous" switch from LV to SV when λ goes from 0 to 1, and that $\forall \lambda_1 \neq \lambda_2$, the models are different.

We note that in the particular Heston case, we can choose:

$$\frac{d\tilde{I}_\lambda(t)}{\tilde{I}_\lambda(t)} = \sqrt{V}_\lambda(t) L_\lambda(t, I_\lambda(t)) dW(t)$$
$$dV_\lambda(t) = \kappa(\theta - V_\lambda(t)) + \lambda\eta\sqrt{V}_\lambda(t) dB(t)$$

We exhibited a parameterization that matches the smile dynamics very roughly, as we only have one parameter to match all YoY option prices for all t_i, K_i^j. We therefore introduce an extension to handle a term-structure on λ: t_i, λ_i:

$$\tilde{a}(t, x, (\lambda_i)_{i\in[1,I]}) = \bar{a}(t, x, \sum_{i=1}^{I} \lambda_i \mathbf{1}_{\{t_{i-1} < t \leq t_i\}})$$
$$\tilde{b}(t, x, (\lambda_i)_{i\in[1,I]}) = \bar{b}(t, x, \sum_{i=1}^{I} \lambda_i \mathbf{1}_{\{t_{i-1} < t \leq t_i\}})$$

4.0.1 Calibration on YoY option prices

The idea is to bootstrap the λ term structure to match the YoY prices for each maturity. Let T_i be the i-th YoY call maturity greater than 1Y (the 1Y YoY calls are exactly the 1Y ZC options, whose prices are fitted with the LV), and $T_0 = 0$. Let $(K_i^j)_{j\in[0,J(i)-1]}$ be the quoted strikes at T_i.

We define u such as $t_{u(i)} = T_i$ and we recall that $(t_i)_{i\in[0,n]}$ are the simulation dates for the particles. The i-th step of the bootstrap is:

1. $(\lambda_j)_{j\leq i-1}$, $(X_j)_{j\leq u(i-1)}$ and $\Sigma(t \leq T_{i-1}, \cdot)$ are known.
2. Find λ_i:
   (a) Choose $\bar{\lambda}_i$.
   (b) Compute $(X_j(\bar{\lambda}_i))_{u(i-1)<j\leq u(i)}$ and $\Sigma(\bar{\lambda}_i)(T_{i-1} < t \leq T_i, \cdot)$ with the particle method.
   (c) $\forall j \in [0, J(i)-1]$ approximate $C_{YoY}(T_i, K_i^j, \bar{\lambda}_i)$ using (3.4).
   (d) Compute
   $$e(\bar{\lambda}_i) = \sum_{j=0}^{J(i)-1} w_{i,j} \left(C_{YoY}(T_i, K_i^j, \bar{\lambda}_i) - C_{YoY}^{mkt}(T_i, K_i^j) \right)^2$$
   (e) Iterate to find $\bar{\lambda}_i$ that minimizes e, and set λ_i to this value.
3. Compute $(X_j)_{u(i-1)<j\leq u(i)}$ and $\Sigma(T_{i-1} < t \leq T_i, \cdot)$ with the particle method.

Note that if the complexity of the particle method is $\mathcal{O}(p(m))$ where m is the number of particle, the complexity of this calibration is $\mathcal{O}((n_\lambda + 1)p(m))$ where n_λ is the number of $\bar{\lambda}$ tried at each date. This complexity may however be reduced.

First, we don't need to use m particles to compute the YoY call prices but can take only $m_\lambda < m$ (for the last part of the algorithm we still use m to be more precise on the leverage surface computation). The complexity is therefore $\mathcal{O}(n_\lambda p(m_\lambda) + p(m))$. A last improvement is to take a small value for n_λ and interpolate $C_{YoY}(T_i, K_i^j, \cdot)$. In our experiments, we took $m_\lambda = m/2$ and $n_\lambda = 3$. More precisely, $\bar{\lambda}_i \in \{\lambda^*_{i-1} - h; \lambda^*_{i-1}; \lambda^*_{i-1} + h\}$ with $\lambda^*_{i-1} = max(\lambda_{i-1}, h)$, and performed a polynomial interpolation.

5 Numerical Results

We implemented the Heston Local Volatility case in LexiFi's pricing library. We consider $m = 10,000$ particles and $m_\lambda = 5,000$ for the λ optimization. We take $\lambda_i \in \{0; \lambda_{i-1}; 2\lambda_{i-1}\}$, $\lambda_0 = 1$. The results were computed on the HICPxT, the UKRPI and the USCPI indices with real data from 2016-02-03 (source: Bloomberg).

5.1 Timing

We compare the total calibration time of the parameters up to ten years between a usual SLV (using particle method) approach and applying our new method. We also give the time of the first Heston Calibration part. All results are presented in Table 1.

We see that calibrating the YoY prices only doubles the calibration time at worst; this is not due to the fact that the first Heston calibration takes most of the time. The described approach combined with a careful implementation that shares some calculations between each λ, e.g. Brownian increments simulation, resulted into this encouraging speed.

Underlying	Heston Calibration	Standard SLV	New SLV	Ratio
HICPxT	1.82 (0.54s)	5.02 (1.48s)	9.58 (2.82s)	1.91
UKRPI	1.44 (0.43s)	4.85 (1.43s)	9.17 (2.70s)	1.89
USCPI	1.33 (0.39s)	4.42 (1.30s)	8.97 (2.64s)	2.03

Table 1: Performances in Mega clock cycles and seconds

Strike	ZC call market price	ZC call calibrated price	ZC call MC price	YoY caplet market price	YoY caplet calibrated price	YoY caplet MC price
1%	92.68	92.46	90.35	60.42	61.86	61.14
2%	18.00	18.02	18.04	25.88	33.37	31.11
3%	5.03	5.11	4.84	12.78	23.72	21.48
4%	2.95	2.99	2.61	7.31	17.93	15.99
5%	1.84	1.78	1.83	4.63	13.79	12.26
6%	1.02	1.03	1.40	3.19	10.69	9.60

Table 2: HICPxT 5Y index call prices and YoY caplet prices, in bps

Strike	Market price	SLV price	SLV MC price	Heston MC price	LV MC price
1%	116.11	116.18	119.57	164.24	211.47
2%	57.78	50.04	53.89	109.11	164.22
3%	35.20	39.08	43.19	79.09	125.44
4%	27.04	32.41	36.37	60.77	94.16
5%	22.62	27.89	31.62	47.93	69.70
6%	19.56	24.53	27.99	38.28	50.91

Table 3: HICPxT 10Y YoY caplet prices in bps

5.2 Fit quality

Table 2 shows the ability of our calibration method to fit almost exactly the Index option prices and to have a good fit of the YoY caplets for the strikes near the forward. Moreover, it validates the polynomial 3 points approximation for computing the YoY caplet prices, as all MC prices are close to the calibrated prices (computed with the approximation). We also compare our approach against pure Heston or pure Local Volatility models calibrated to index options. Table 3 shows that even if we use a model that has a good (or perfect) fit of the index options, we don't have necessarily a good fit of the YoY caplets, and how the SLV model improved the YoY calibration. It shows also that the magnitude of errors are not that big (compared to other models) on both tests.

6 Extension to stochastic rates

In this section, we present a possible extension of the calibration method to handle stochastic rate. The model becomes:

$$\begin{aligned}\frac{dI(t)}{I(t)} &= [n(t) - r(t)]dt + \sigma(t)L(t, I(t))dW(t) \\ d\sigma(t) &= a(t, \sigma(t))dt + b(t, \sigma(t))dB(t) \\ dW(t)dB(t) &= \rho dt\end{aligned}$$

where n and r are $(\mathcal{F}_t)$-measurable processes.

6.1 Leverage Surface Computation

Extending the calibration implies first to derive the leverage surface expression in the case of stochastic interest rates; we introduce the process I_0 defined by

$$
\begin{aligned}
\frac{dI_0(t)}{I_0(t)} &= [n_0(t) - r_0(t)]dt + \Sigma(t, I_0(t))dW(t) \\
n_0(t) &= \mathbb{E}^t\left[n(t)\right] \\
r_0(t) &= \mathbb{E}^t\left[r(t)\right]
\end{aligned}
$$

and build it so that it matches also the option prices (it has therefore the same distribution as I at each t). This can be achieved by setting Σ thanks to the Dupire equation [Dup94].

We set $D(t) := e^{-\int_0^t n(u)du}$ so that

$$C(t, K) = \mathbb{E}[D(t)(I(t) - K)^+]$$

Straightforward computations lead to:

$$d\left[D(t)(I(t) - K)^+\right] = [n(t)K - r(t)I(t)]D(t)\mathbf{1}_{I(t)>K}dt + \frac{1}{2}D(t)\delta(I(t) - K)\sigma(t)^2 L(t, I(t))^2 I(t)^2 dt + \mathcal{O}\left(dW(t)\right)$$

We take the nominal t-forward expectation and divide it by dt:

$$\partial_t C(t, K) = P^n(0, t)\mathbb{E}^t\left[[n(t)K - r(t)I(t)]\mathbf{1}_{I(t)>K}dt\right] + \frac{1}{2}L(t, K)^2 K^2 \mathbb{E}^t\left[\sigma(t)^2 | I(t) = K\right]\partial_K^2 C(t, K)$$

Similarly, and building I_0 so that it also matches the call prices (having therefore the same distribution as I), we get:

$$
\begin{aligned}
\partial_t C(t, K) &= P^n(0, t)\mathbb{E}^t\left[[n_0(t)K - r_0(t)I_0(t)]\mathbf{1}_{I_0(t)>K}dt\right] + \frac{1}{2}\Sigma(t, K)^2 K^2 \partial_K^2 C(t, K) \\
&= P^n(0, t)\mathbb{E}^t\left[[n_0(t)K - r_0(t)I(t)]\mathbf{1}_{I(t)>K}dt\right] + \frac{1}{2}\Sigma(t, K)^2 K^2 \partial_K^2 C(t, K)
\end{aligned}
$$

Then:

$$L(t, K)^2 = \left(\Sigma(t, K)^2 + 2P^n(0, t)\frac{\mathbb{E}^t\left[[(n_0(t) - n(t))K - (r_0(t) - r(t))I(t)]\mathbf{1}_{I(t)>K}dt\right]}{K^2 \partial_K^2 C(t, K)}\right)\frac{1}{\mathbb{E}^t\left[\sigma(t)^2 | I(t) = K\right]}$$

Expressed under the nominal T-forward measure:

$$
\begin{aligned}
L(t, K)^2 = &\left(\Sigma(t, K)^2 + 2P^n(0, T)\frac{\mathbb{E}^T\left[P^n(t, T)^{-1}[(n_0(t) - n(t))K - (r_0(t) - r(t))I(t)]\mathbf{1}_{I(t)>K}dt\right]}{K^2 \partial_K^2 C(t, K)}\right) \\
&\times \frac{\mathbb{E}^T\left[P^n(t, T)^{-1} | I(t) = K\right]}{\mathbb{E}^T\left[P^n(t, T)^{-1}\sigma(t)^2 | I(t) = K\right]}
\end{aligned}
\tag{6.1}
$$

We note that given C, we can compute the local volatility surface Σ of I_0 and apply the particle method to compute the leverage surface L, assuming that we can simulate I, n, r and $P^n(\cdot, T)$ in the nominal T-forward measure (possible in a HJM short-rate model for instance).

6.2 Note on the index simulation

In this section, we show how to simulate the index under the nominal T-forward measure, whatever the assumptions on the interest rates. First, using the FX analogy, we have:

$$P^n(t,T)F(t,T) = P^r(t,T)I(t)$$

Then, under the nominal T-forward measure, the dynamics of S can be deduced from the one of F:

$$\begin{aligned}\frac{dF(t,T)}{F(t,T)} &= \sigma(t)L(t,I(t))dW^T(t)\\ I(t) &= \frac{P^n(t,T)}{P^r(t,T)}F(t,T)\end{aligned}$$

This gives a generic way to simulate I under the required measure.

6.3 YoY option prices

To apply our method, we need to be able to compute the YoY option prices within the particle method, or equivalently to be able to express these prices under the nominal T-forward measure, for all maturity t. We simply have to apply the pricing formula in the needed numeraire:

$$C_{YoY}(t,K) = P^n(0,T)\mathbb{E}^T\left[P^n(t,T)^{-1}\left(\frac{I(t)}{I(t-1)} - 1 - K\right)^+\right] \tag{6.2}$$

Then, our calibration method applies, using the adapted formulation of the leverage surface (6.1), and the YoY price formula under the nominal T-forward measure (6.2).

6.4 Interest rate calibration

We present a simple method to calibrate the short rates, even if it could probably be improved. We choose a classical Hull-White model for the short rates:

$$\begin{aligned}dn(t) &= [\theta_n(t) - a_n n(t)]\,dt + \sigma_n dW_n(t)\\ dr(t) &= [\theta_r(t) - a_r r(t) - \rho_{r,I}\sigma_r\sigma(t)L(t,I(t))]\,dt + \sigma_r dW_r(t)\\ dW_n(t)dW(t) &= \rho_{n,I}dt\\ dW_r(t)dW(t) &= \rho_{r,I}dt\\ dW_n(t)dW_r(t) &= \rho_{n,r}dt\end{aligned}$$

under the nominal cash measure.

It is easy to express the dynamics of n and r under the nominal T-forward measure, using standard results of the

Hull-White model. For instance, using [AP10], section 10.1.6.3:

$$
\begin{aligned}
dn(t) &= \left[\theta_n(t) - \sigma_n^2 G(t,T) - a_n n(t)\right] dt + \sigma_n dW_n^T(t) \\
dr(t) &= \left[\theta_r(t) - \rho_{n,r}\sigma_n\sigma_r G(t,T) - a_r r(t) - \rho_{r,I}\sigma_r\sigma(t)L(t,I(t))\right] dt + \sigma_r dW_r^T(t) \\
G(t,T) &= \frac{1 - e^{-a_n(T-t)}}{a_n}
\end{aligned}
$$

The short rate parameters a_n, a_r, σ_n, σ_r, $\rho_{n,I}$, $\rho_{r,I}$ and $\rho_{n,r}$ can be taken from a simple Jarrow-Yildirim model calibration. This model has been studied in [Mer05] for instance.

To summarize, the calibration of the SLV-HW model for inflation resumes to:

1. Calibrate the short rate parameters on a Jarrow-Yildirim model.
2. Apply our calibration method with the adapted formulation of the leverage surface (6.1), and pricing the YoY option under the nominal T-forward measure (6.2).

7 Conclusion

We presented a calibration method of stochastic volatility models for inflation modeling, based on the recent particle method, that allows to perfectly fit the zero coupon option prices (the smile) while achieving also a good fit of the YoY option prices (the smile dynamics). We showed that the calibration is fast regarding the complexity of the problem and therefore of practical relevance (it is now distributed as part of LexiFi's integrated quantitative library).

Another benefit of this approach is the small number of factors (two), compared to an Inflation Market Model.

This approach can also be an efficient alternative to calibrate a SLV model for FX, if one has access to some smile dynamics quotes, e.g. one touch option prices (we recall that the instruments to calibrate the mixing fraction can be path-dependent).

We also sketched a possible extension of the method to stochastic interest rates, a feature that should be useful for pricing long dated Inflation (or FX) contracts.

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